

Homework 11 Solutions

Math 121A, Fall 2025.

Problem 1. Many examples.

(a) The function $f(x) = 1$ is in $L^2([-\pi, \pi])$ since

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \, dx = 1 < \infty.$$

(b) The function $f(x) = \frac{1}{x}$ is not in $L^2([-\pi, \pi])$, since

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{x^2} \, dx = \frac{1}{\pi} \int_0^{\pi} \frac{1}{x^2} \, dx = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left(-\frac{1}{\epsilon} + \frac{1}{\pi}\right) = \infty.$$

(c) The function $f(x) = \frac{1}{|x|^{1/4}}$ is in $L^2([-\pi, \pi])$ since

$$\|f\|^2 = \frac{1}{\pi} \int_0^{\pi} \frac{1}{\sqrt{x}} \, dx = \frac{1}{\pi} (2\sqrt{\pi} - 2\sqrt{0}) < \infty.$$

(d) The sequence $\delta_0 : \mathbb{Z} \rightarrow \mathbb{C}$ given by

$$\delta_0(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

is in $\ell^2(\mathbb{Z})$ since

$$\|\delta_0\|^2 = \sum_{k=-\infty}^{\infty} |\delta_0(k)|^2 = 1^2 < \infty.$$

(e) The constant sequence $s(k) = 1$ for all $k \in \mathbb{Z}$ is not in $\ell^2(\mathbb{Z})$, since

$$\|s\|^2 = \sum_{k=-\infty}^{\infty} |s(k)|^2 = \sum_{k=-\infty}^{\infty} 1 = \infty.$$

Problem 2. (*Simultaneous diagonalization and Heisenberg uncertainty*, $\dim \mathcal{H} = n$)

Consider the following pair of matrices, expressed in the standard basis of $\mathbb{C}^2$:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Suppose first that A, B are simultaneously diagonalizable. Thus, there is shared basis of eigenvectors $\{v_1, \dots, v_n\}$ of these operators, say with respective eigenvalues λ_i for A and μ_i for B . Observe that for each $i = 1, \dots, n$,

$$ABv_i = A\mu_i v_i = \mu_i Av_i = \mu_i \lambda_i v_i = \lambda_i Bv_i = BAv_i.$$

Since this is true on a basis, it must be that $AB = BA$.

Now, suppose that $AB = BA$. We first show that B respects the eigenspaces of A , meaning if v is an eigenvector of eigenvalue λ , then so is Bv :

$$A(Bv) = BAv = B\lambda v = \lambda(Bv).$$

Now, if the dimension of each eigenspace $V_\lambda := \{v \in \mathbb{C}^n : Av = \lambda v\}$ were 1, then we would be done since this would mean each v is an eigenvector of B . So, consider the situation where $\dim(V_\lambda) > 1$. If we restrict to this eigenspace, observe that A acts as $\lambda\mathbb{1}$, since $Av = \lambda\mathbb{1}v$ for all $v \in V_\lambda$. Since B is diagonalizable on $\mathbb{C}^n$, it is diagonalizable on V_λ , and so there is a change of basis matrix $S : V_\lambda \rightarrow V_\lambda$ such that $SBS^{-1} = D$, where D is a diagonal matrix on V_λ . Applying this change of basis to $A = \lambda\mathbb{1}$, we get back $SAS^{-1} = \lambda SS^{-1} = \lambda\mathbb{1}$. In particular, A and B are simultaneously diagonalized on V_λ . By applying the same procedure to any other eigenspace V_{μ} , we end up with A and B simultaneously diagonalized.

- (b) This is just a quick calculation:

$$[X, Z] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \neq 0$$

This means that X and Z cannot be simultaneously diagonalized.

- (c) Off we go. The eigenvalues of X are $1, -1$ with respective eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. So,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = UDU^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

U is certainly unitary because its columns form an orthonormal basis of $\mathbb{C}^2$.

- (d) To change basis, we write

$$U^{-1}ZU = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definitely not diagonal in this basis.

Problem 3. (*Simultaneous diagonalization and Heisenberg uncertainty, $\dim \mathcal{H} = \infty$*)

- (a) Let's compute, first fixing a (differentiable) $f \in L^2([-\pi, \pi])$:

$$[X, P](f) = (XP - PX)(f) = X(-i\frac{d}{dx}f) + i\frac{d}{dx}(xf) = -ix\frac{df}{dx} + i(f + x\frac{df}{dx}) = if.$$

So, the commutator of these operators is nonzero (the above shows that $[X, P] = i\mathbb{1}$), and thus they do not commute and cannot be simultaneously diagonalized.

- (b) We actually computed these Fourier series coefficients in class, but here they are again:

$$\widehat{f_m}(x) = \widehat{e^{imx}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix(m-k)} dx = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{otherwise.} \end{cases}$$

So in particular, $\widehat{f_m}(k) = \delta_m(k)$. We stated in class that the collection $\{\delta_m : m \in \mathbb{Z}\}$ forms an orthonormal basis of $\ell^2(\mathbb{Z})$. So yes, the map $\hat{\cdot}$ sends an orthonormal basis to an orthonormal basis.

- (c) We showed this one in class too, but it's worth repeating!

This was the calculation that

$$\widehat{-i \frac{d}{dx} f(k)} = k \widehat{f(k)}.$$

In particular, $-i \frac{d}{dx}$ is not a multiplication operator (“not diagonal”), but k is a multiplication operator (“is diagonal”). We can also use the notation with $\mathcal{F} : L^2([-\pi, \pi]) \rightarrow \ell^2(\mathbb{Z})$ given by $\mathcal{F}(f) = \hat{f}$. Then the above equation can be re-written as

$$\mathcal{F}(-i \frac{d}{dx} f) = k \mathcal{F}(f),$$

whence by taking inverse fourier transforms of both sides,

$$-i \frac{d}{dx} f = \mathcal{F}^{-1} k \mathcal{F}(f).$$

This is of the form $P = \mathcal{F}^{-1} k \mathcal{F}$, with $\mathcal{F}$ unitary and k diagonal.

Problem 4. (*The Discrete Fourier Transform*)

- (a) First, the translation operator:

$$[S] = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Now, the discrete Fourier transform. As the problem suggests we introduce the number $\omega = e^{-i2\pi/n}$

$$[U] = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{(n-1)2} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix}.$$

The $(x, k)^{th}$ entry is

$$(U)_{x,k} = \omega^{x \cdot k} = e^{\frac{-i2\pi}{n} xk}, \quad x, k \in \{0, 1, \dots, n-1\}.$$

- (b) To verify that U is unitary, we need only check that it has orthonormal columns. First, we check normalization: fix a column $x \in \{0, 1, \dots, n-1\}$. Then its norm is

$$\frac{1}{n} \sum_{k=0}^{n-1} \bar{\omega}^{xk} \omega^{xk} = \frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi xk/n} e^{-i2\pi xk/n} = \frac{1}{n} \sum_{k=0}^{n-1} 1 = n/n = 1,$$

which is what we wanted. Now, if we take two different columns $x \neq y$, then their dot product is

$$\frac{1}{n} \sum_{k=0}^{n-1} \bar{\omega}^{xk} \omega^{yk} = \frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi yk/n} e^{-i2\pi xk/n} = \frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi k(x-y)/n}.$$

We then remember the trick. Since $x \neq y$, the number $z := e^{i2\pi k(x-y)/n}$ is not 1. Thus the above becomes

$$\frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi k(x-y)/n} = \frac{1}{n} \sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} = \frac{1-1}{1-z} = 0,$$

where we have used that $z^n = 1$. Thus, distinct columns are orthogonal, completing the proof that U is unitary.

- (c) Let's calculate. Since U is unitary, we have $U^{-1} = U^\dagger$ and so for $n = 2$ we have

$$[USU^{-1}] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Yes, the same as in the Heisenberg uncertainty problem! and for $n = 3$ we have

$$[USU^{-1}] = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

More generally, we obtain that the eigenvalues of UDU^{-1} are exactly the collection $\{\omega^k = e^{-i2\pi k/n} : k = 0, \dots, n-1\}$.

- (d) The relation is exactly that $\partial = \mathbb{1} - S$. Then, we have

$$[U\partial U^\dagger] = [U(\mathbb{1} - S)U^\dagger] = [UU^\dagger] - [USU^\dagger] = [\mathbb{1}] - [USU^\dagger].$$

Since $[\mathbb{1}]$ and $[USU^\dagger]$ are both diagonal, their difference is still diagonal and we are done.

Problem 5. (*Spectrum and operators*)

Let $\mathcal{H}$ be a Hilbert space.

- (a) $A = A^\dagger$. Let λ be an eigenvalue of A with eigenvector v , say $\|v\| = 1$. Then

$$\lambda \|v\|^2 = \lambda \langle v, v \rangle = \langle v, \lambda v \rangle = \langle v, Av \rangle = \langle A^\dagger v, v \rangle = \langle Av, v \rangle = \langle \lambda v, v \rangle = \bar{\lambda} \|v\|^2.$$

So, dividing over $\|v\|^2$, we have $\lambda = \bar{\lambda}$, so it is a real number.

- (b) Take an eigenvalue λ of P with eigenvector v . Then

$$\lambda v = Pv = P^2v = \lambda^2 v.$$

Subtracting over, this implies that $\lambda^2 = \lambda$, which is only true if λ is either 0 or 1.

- (c) Take an eigenvalue λ of U with eigenvector v . Then observe that

$$|\lambda| \|v\| = \|\lambda v\| = \|Uv\| = \|v\|.$$

Dividing both sides by $\|v\|$, we have $|\lambda| = 1$, i.e. λ is on the unit circle.

Problem 6. (*Examples from integral operators*)

A very useful class of linear operators comes from integral operators $I_K : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$, which are defined by taking a function $K : [-\pi, \pi] \times [-\pi, \pi] \rightarrow \mathbb{C}$ and writing

$$(I_K(f))(x) = \int_{-\pi}^{\pi} K(x, y) f(y) dy \quad \text{for all } f \in L^2([-\pi, \pi]).$$

- (a) Let us show that I_K is self-adjoint when $K(x, y) = |x - y|$. Let $f, g \in L^2([-\pi, \pi])$. Then by Fubini's theorem, we can exchange the order of integration to get

$$\begin{aligned}\langle f, I_K g \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \int_{-\pi}^{\pi} K(x, y) g(y) dy dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\overline{K(x, y) f(x)} \right) g(y) dy dx \\ &= \langle I_K f, g \rangle.\end{aligned}$$

- (b) We just need to show that $I_K^2 = I_K^\dagger = I_K$, and that $I_K \psi(x) = \psi(x)$ and $I_K f = 0$ whenever $\langle f, \psi \rangle = 0$.

Off we go.

$$\begin{aligned}\langle f, I_K g \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \int_{-\pi}^{\pi} \psi(x) \overline{\psi(y)} g(y) dy dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\overline{\psi(x) \psi(y) f(x)} \right) g(y) dy dx \\ &= \langle I_K f, g \rangle.\end{aligned}$$

This shows that $I_K = I_K^\dagger$. Now we need to show that $I_K^2 = I_K$.

$$(I_K^2 f)(x) = \int_{-\pi}^{\pi} \psi(x) \overline{\psi(y)} \left(\int_{-\pi}^{\pi} \psi(y) \overline{\psi(z)} f(z) dz \right) dy = \int_{-\pi}^{\pi} \psi(x) \left(\int_{-\pi}^{\pi} \overline{\psi(y)} \psi(y) dy \right) f(x) dx = \|\psi\|^2 (I_K f)(x).$$

Since $\|\psi\| = 1$, we are done.

The last pair of things to show is that $I_K \psi(x) = \psi(x)$ and if f is orthogonal to ψ , $\langle \psi, f \rangle = 0$, then $I_K f = 0$. Let's do it in one shot:

$$(I_K f)(x) = \int_{-\pi}^{\pi} \psi(x) \overline{\psi(y)} f(y) dy = 2\pi \psi(x) \langle \psi, f \rangle.$$

From here the desired result is apparent: the right hand side is $\psi(x)$ if $f = \psi$, since $\langle \psi, \psi \rangle = 1$, and it is 0 if f is orthogonal to ψ since $\langle \psi, f \rangle = 0$.

Problem 7. (*Translations and Derivatives*)

- (a) Let $f \in L^2([-\pi, \pi])$. Then S_θ is invertible with inverse $\tau_{-\theta}$, since $(\tau_\theta \tau_{-\theta} f)(x) = f(x - \theta + \theta) = f(x)$. It also preserves norms, and so must be unitary, since

$$\|S_\theta f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(\tau_\theta f)(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x - \theta)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 dx = \|f\|^2,$$

where the mod 2π business allowed us to easily change variables $x \mapsto x - \theta$.

- (b) Let's compute.

$$\begin{aligned}\langle f, \partial_\theta g \rangle &= \lim_{\theta \rightarrow 0} \frac{1}{\theta} \langle f, g - S_\theta g \rangle = \lim_{\theta \rightarrow 0} \frac{1}{\theta} (\langle f, g \rangle - \langle f, S_\theta g \rangle) = \lim_{\theta \rightarrow 0} \frac{1}{\theta} (\langle f, g \rangle - \langle \tau_{-\theta} f, g \rangle) \\ &= \lim_{\theta \rightarrow 0} \frac{1}{\theta} \langle f - \tau_{-\theta} f, g \rangle \\ &= \lim_{-\theta \rightarrow 0} \frac{1}{-\theta} \langle f - \tau_\theta f, g \rangle \\ &= \langle (-\partial_\theta) f, g \rangle,\end{aligned}$$

where in the third line we have exchanged the limit as $\theta \rightarrow 0$ with the equivalent $-\theta \rightarrow 0$.